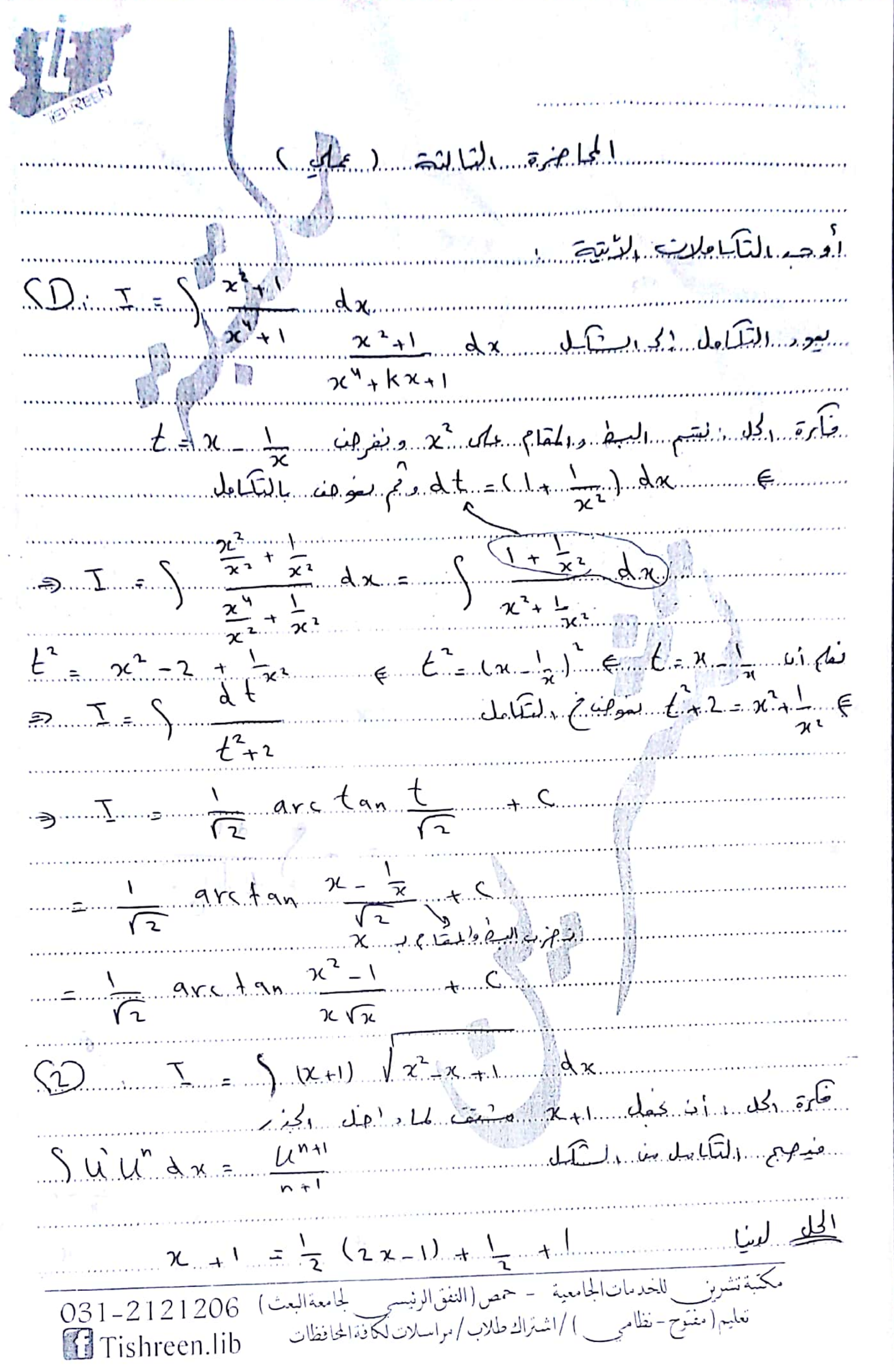


المحاضرة الثالثة (عملي)

أوجد التكاملات الآتية :

(1) : $I = \int \frac{x^2+1}{x^4+1} dx$

يعود التكامل إلى الشكل $\frac{x^2+1}{x^4+kx+1} dx$

فكرة الحل : نقسم البسط والمقام على x^2 ونفرض $t = x - \frac{1}{x}$

⇐ $dt = (1+\frac{1}{x^2})dx$ ثم نعوض بالتكامل

$$\Rightarrow I = \int \frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^4}{x^2}+\frac{1}{x^2}}dx = \int \frac{(1+\frac{1}{x^2})dx}{x^2+\frac{1}{x^2}}$$

نعلم أن $t = x - \frac{1}{x}$ ⇐ $t^2 = (x-\frac{1}{x})^2$ ⇐ $t^2 = x^2 - 2 + \frac{1}{x^2}$

⇐ $t^2+2 = x^2+\frac{1}{x^2}$ نعوض بالتكامل

$$\Rightarrow I = \int \frac{dt}{t^2+2}$$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} + c$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}+c$$

نضرب البسط والمقام بـ x

$$= \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{x}}+c$$

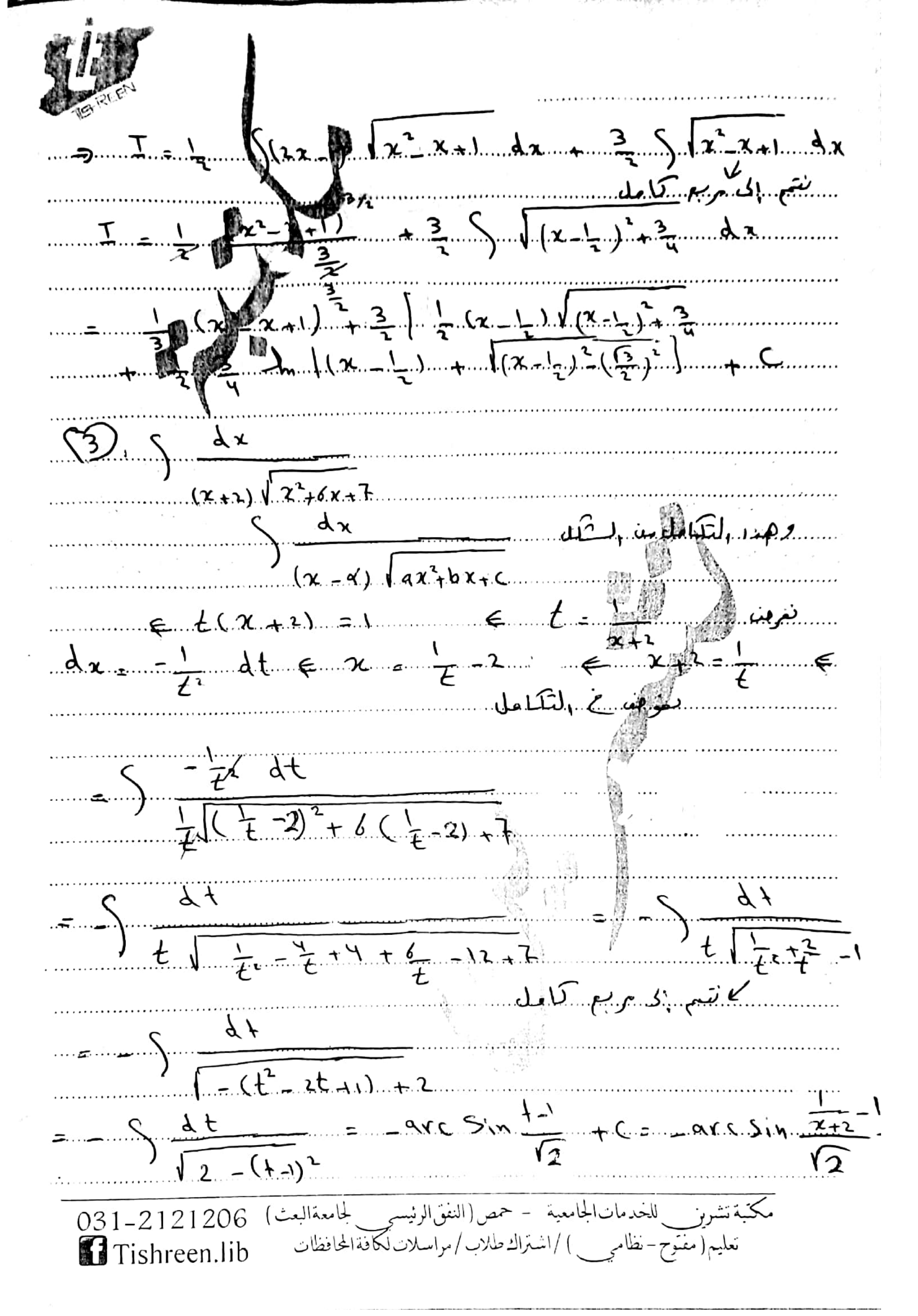
(2) : $I = \int (x+1)\sqrt{x^2-x+1}\,dx$

فكرة الحل : أن نجعل $x+1$ مشتق لما داخل الجذر

فيصبح التكامل من الشكل $\int u' u^n dx = \frac{u^{n+1}}{n+1}$

الحل : لدينا

$$x+1 = \frac{1}{2}(2x-1)+\frac{1}{2}+1$$



$$\Rightarrow I = \frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}\,dx + \frac{3}{2}\int \sqrt{x^2-x+1}\,dx$$

نتمم إلى مربع كامل

$$I = \frac{1}{2}\frac{(x^2-x+1)^{3/2}}{\frac{3}{2}} + \frac{3}{2}\int \sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}(x^2-x+1)^{\frac{3}{2}} + \frac{3}{2}\Big[\frac{1}{2}(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}$$
$$+ \frac{3}{2}\cdot\frac{3}{4}\ln\Big|(x-\frac{1}{2}) + \sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\Big|\Big] + C$$

(3) $\int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

وهذا التكامل من الشكل $\int \frac{dx}{(x-\alpha)\sqrt{ax^2+bx+c}}$

نفرض $t = \frac{1}{x+2} \Leftarrow t(x+2)=1$

$\Leftarrow x+2 = \frac{1}{t} \Leftarrow x = \frac{1}{t}-2 \Leftarrow dx = -\frac{1}{t^2}\,dt$

نعوض في التكامل

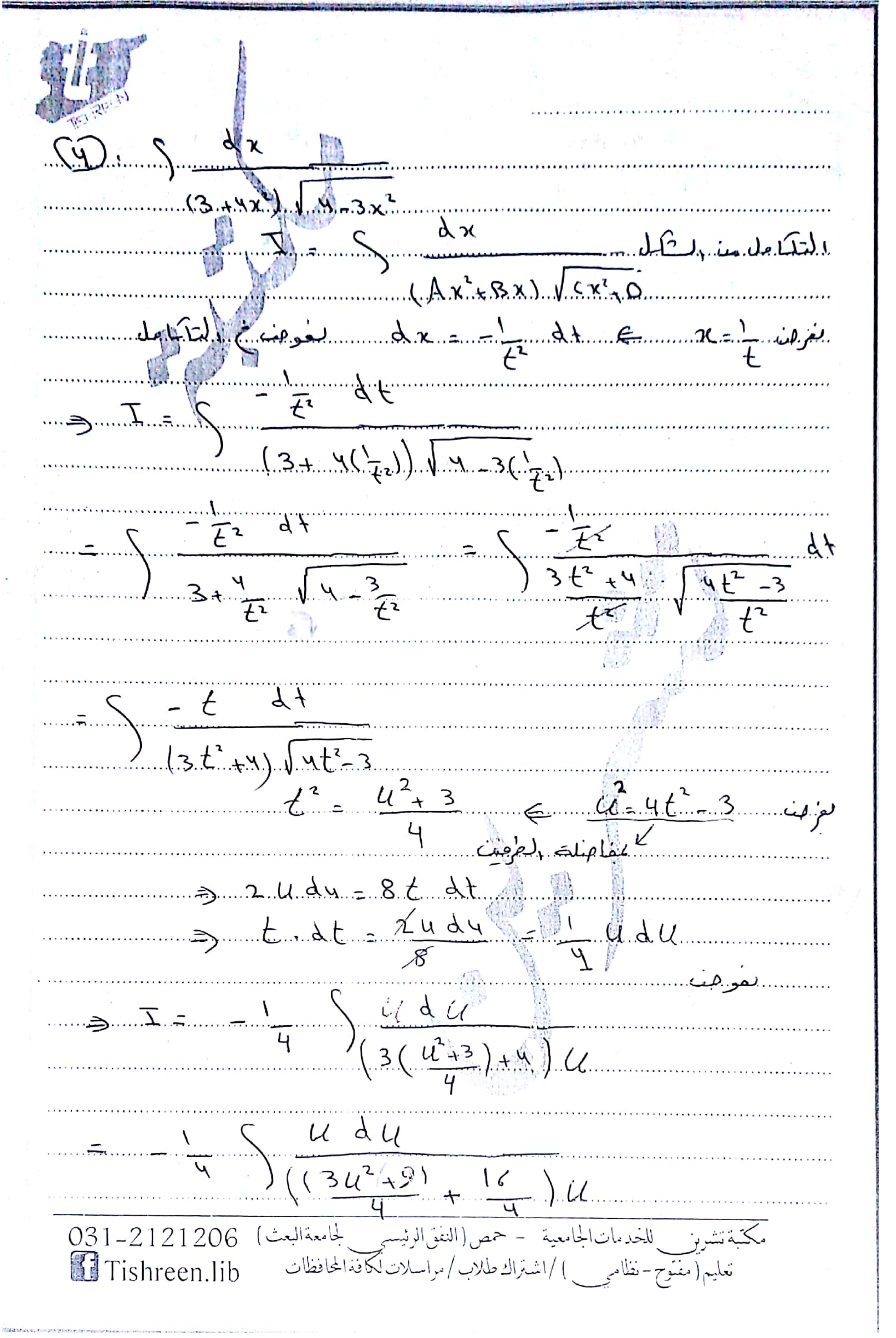
$$= \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{(\frac{1}{t}-2)^2+6(\frac{1}{t}-2)+7}}$$

$$= -\int \frac{dt}{t\sqrt{\frac{1}{t^2}-\frac{4}{t}+4+\frac{6}{t}-12+7}} = -\int \frac{dt}{t\sqrt{\frac{1}{t^2}+\frac{2}{t}-1}}$$

نتمم إلى مربع كامل

$$= -\int \frac{dt}{\sqrt{-(t^2-2t+1)+2}}$$

$$= -\int \frac{dt}{\sqrt{2-(t-1)^2}} = -\arcsin\frac{t-1}{\sqrt{2}} + C = -\arcsin\frac{\frac{1}{x+2}-1}{\sqrt{2}}$$



(4). $\int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$

التكامل من الشكل $I = \int \frac{dx}{(Ax^2+Bx)\sqrt{Cx^2+D}}$

نفرض $x = \frac{1}{t} \Leftarrow dx = -\frac{1}{t^2}\, dt$ نعوض في التكامل

$\Rightarrow I = \int \frac{-\frac{1}{t^2}\, dt}{(3+4(\frac{1}{t^2}))\sqrt{4-3(\frac{1}{t^2})}}$

$= \int \frac{-\frac{1}{t^2}\, dt}{3+\frac{4}{t^2}\sqrt{4-\frac{3}{t^2}}} = \int \frac{-\frac{1}{t^2}}{\frac{3t^2+4}{t^2}\cdot\sqrt{\frac{4t^2-3}{t^2}}}\, dt$

$= \int \frac{-t\, dt}{(3t^2+4)\sqrt{4t^2-3}}$

نفرض $u^2 = 4t^2-3 \Rightarrow t^2 = \frac{u^2+3}{4}$

نفاضل الطرفين

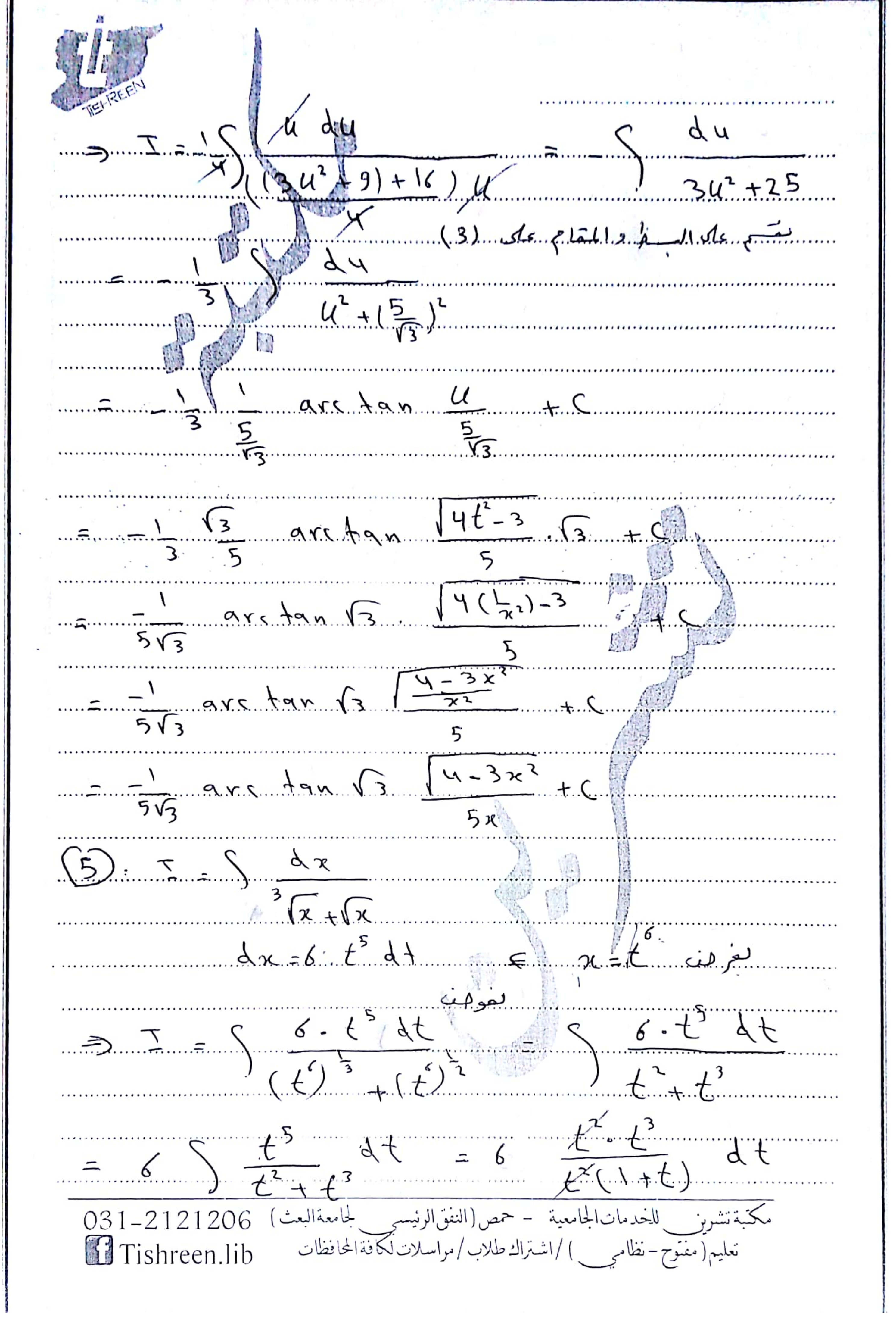
$\Rightarrow 2u\,du = 8t\,dt$

$\Rightarrow t\,dt = \frac{2u\,du}{8} = \frac{1}{4}u\,du$

نعوض

$\Rightarrow I = -\frac{1}{4}\int \frac{u\,du}{(3(\frac{u^2+3}{4})+4)u}$

$= -\frac{1}{4}\int \frac{u\,du}{((\frac{3u^2+9}{4})+\frac{16}{4})u}$



$$\Rightarrow I = \frac{1}{4}\int \frac{u\,du}{\frac{((3u^2+9)+16)}{4}\,u} = -\int \frac{du}{3u^2+25}$$

نقسم على البسط والمقام على (3)

$$= -\frac{1}{3}\int \frac{du}{u^2+\left(\frac{5}{\sqrt{3}}\right)^2}$$

$$= -\frac{1}{3}\,\frac{1}{\frac{5}{\sqrt{3}}}\arctan\frac{u}{\frac{5}{\sqrt{3}}} + c$$

$$= -\frac{1}{3}\,\frac{\sqrt{3}}{5}\arctan\frac{\sqrt{4t^2-3}\cdot\sqrt{3}}{5} + c$$

$$= \frac{-1}{5\sqrt{3}}\arctan\sqrt{3}\cdot\frac{\sqrt{4\left(\frac{1}{x^2}\right)-3}}{5} + c$$

$$= \frac{-1}{5\sqrt{3}}\arctan\sqrt{3}\,\frac{\sqrt{\frac{4-3x^2}{x^2}}}{5} + c$$

$$= \frac{-1}{5\sqrt{3}}\arctan\sqrt{3}\,\frac{\sqrt{4-3x^2}}{5x} + c$$

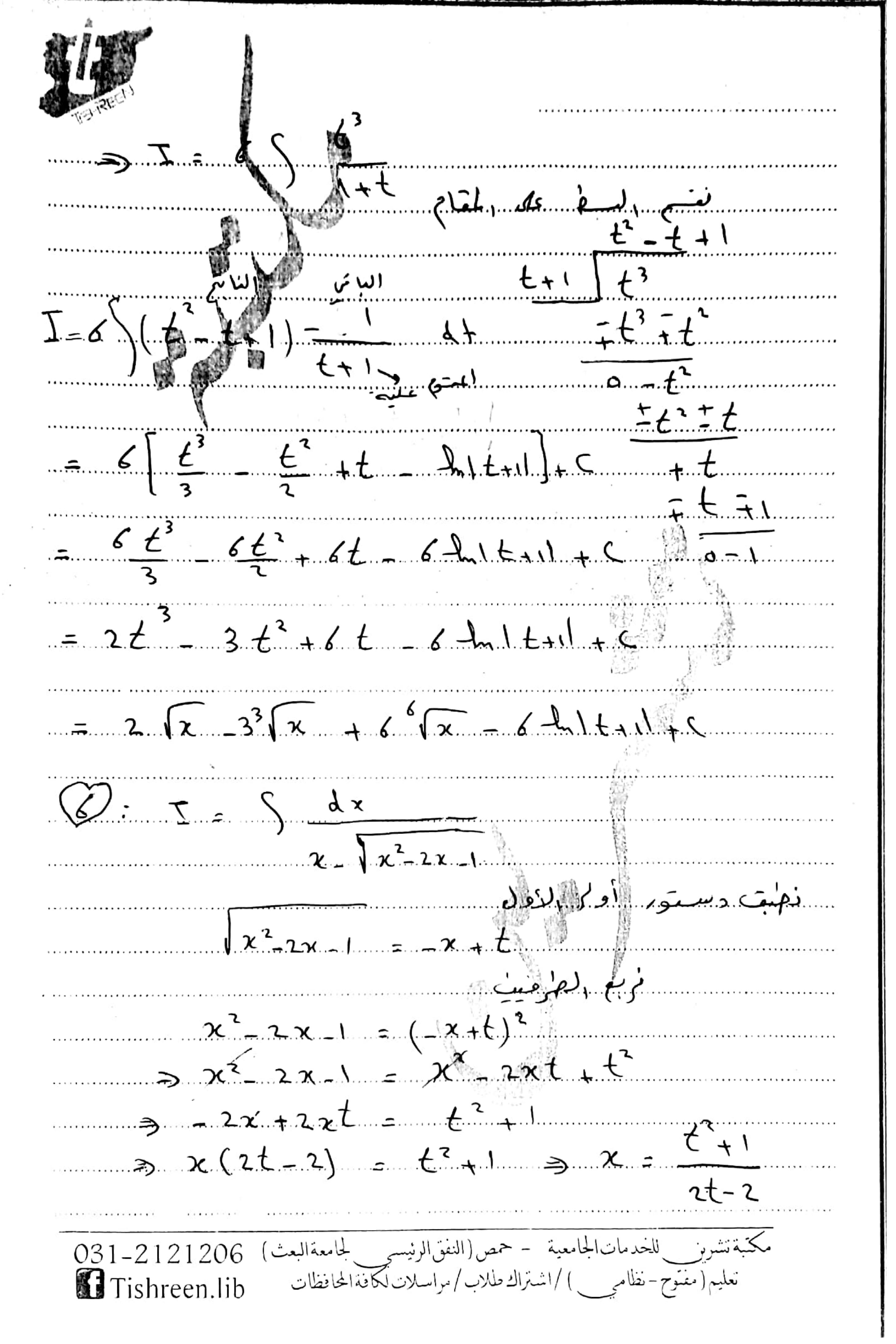
(5): $I = \int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}$

نفرض $x = t^6 \Leftarrow dx = 6\,t^5\,dt$

نعوض

$$\Rightarrow I = \int \frac{6\cdot t^5\,dt}{(t^6)^{\frac{1}{3}}+(t^6)^{\frac{1}{2}}} = \int \frac{6\cdot t^5\,dt}{t^2+t^3}$$

$$= 6\int \frac{t^5}{t^2+t^3}\,dt = 6\int \frac{t^2\cdot t^3}{t^2(1+t)}\,dt$$



$$\Rightarrow I = 6\int \frac{t^3}{1+t}$$

نقسم البسط على المقام

$$\begin{array}{r|l} & t^2 - t + 1 \\ t+1 & t^3 \\ & \mp t^3 \mp t^2 \\ & 0 - t^2 \\ & \pm t^2 \pm t \\ & + t \\ & \mp t \mp 1 \\ & 0 - 1 \end{array}$$

الناتج، الباقي، المقسوم عليه

$$I = 6\int \left((t^2 - t + 1) - \frac{1}{t+1} \right) dt$$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1| \right] + c$$

$$= \frac{6t^3}{3} - \frac{6t^2}{2} + 6t - 6\ln|t+1| + c$$

$$= 2t^3 - 3t^2 + 6t - 6\ln|t+1| + c$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|t+1| + c$$

⑥: $$I = \int \frac{dx}{x - \sqrt{x^2 - 2x - 1}}$$

نطبق دستور أولر الأول

$$\sqrt{x^2 - 2x - 1} = -x + t$$

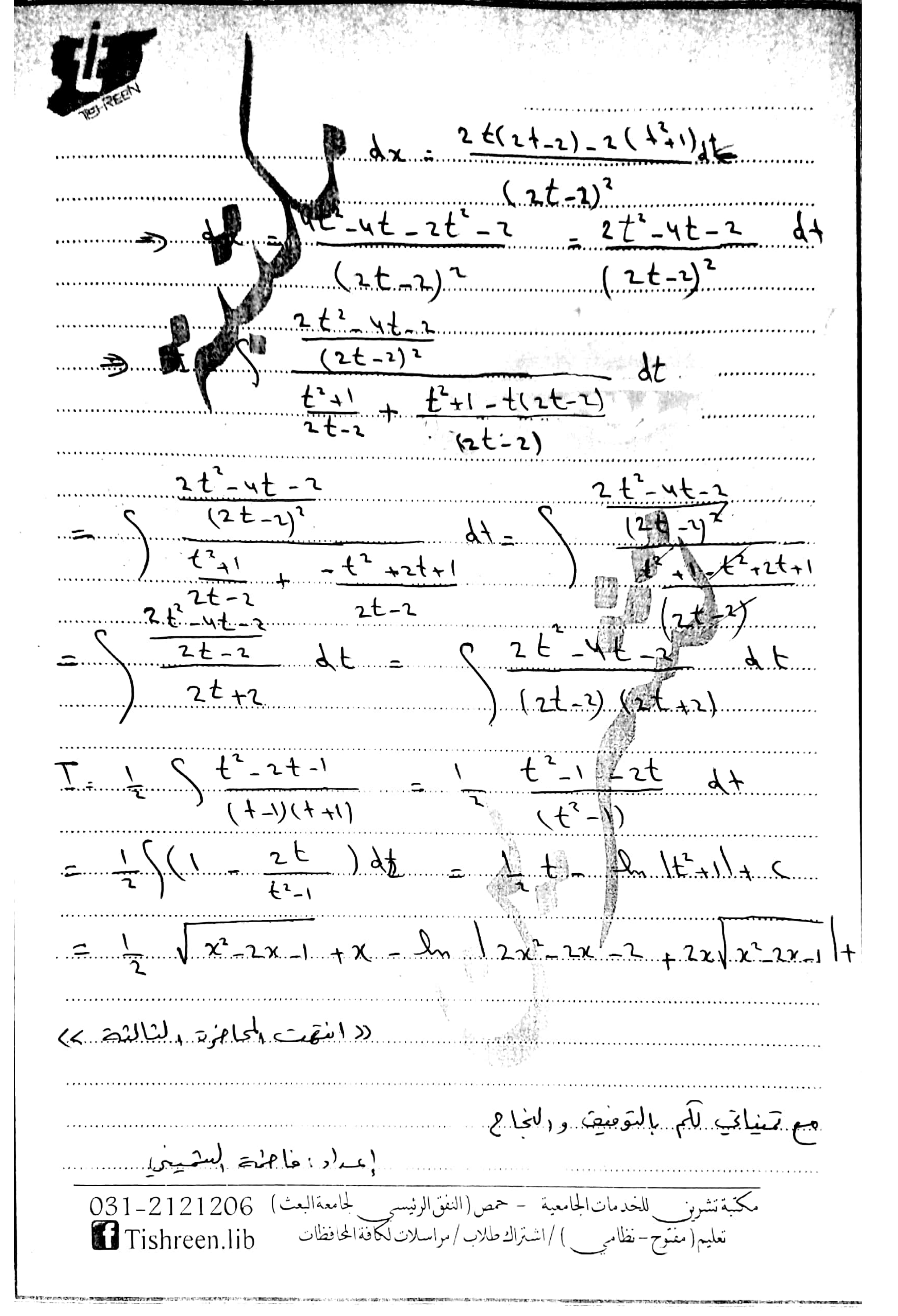
نربع الطرفين

$$x^2 - 2x - 1 = (-x + t)^2$$

$$\Rightarrow x^2 - 2x - 1 = x^2 - 2xt + t^2$$

$$\Rightarrow -2x + 2xt = t^2 + 1$$

$$\Rightarrow x(2t - 2) = t^2 + 1 \Rightarrow x = \frac{t^2 + 1}{2t - 2}$$



$$dx = \frac{2t(2t-2) - 2(t^2+1)}{(2t-2)^2}\,dt$$

$$\Rightarrow dx = \frac{4t^2 - 4t - 2t^2 - 2}{(2t-2)^2} = \frac{2t^2 - 4t - 2}{(2t-2)^2}\,dt$$

$$\Rightarrow I = \int \frac{\frac{2t^2-4t-2}{(2t-2)^2}}{\frac{t^2+1}{2t-2} + \frac{t^2+1-t(2t-2)}{(2t-2)}}\,dt$$

$$= \int \frac{\frac{2t^2-4t-2}{(2t-2)^2}}{\frac{t^2+1}{2t-2} + \frac{-t^2+2t+1}{2t-2}}\,dt = \int \frac{\frac{2t^2-4t-2}{(2t-2)^2}}{\frac{t^2+1-t^2+2t+1}{(2t-2)}}$$

$$= \int \frac{\frac{2t^2-4t-2}{2t-2}}{2t+2}\,dt = \int \frac{2t^2-4t-2}{(2t-2)(2t+2)}\,dt$$

$$I = \frac{1}{2} \int \frac{t^2-2t-1}{(t-1)(t+1)} = \frac{1}{2} \int \frac{t^2-1-2t}{(t^2-1)}\,dt$$

$$= \frac{1}{2} \int \left(1 - \frac{2t}{t^2-1}\right) dt = \frac{1}{2}t - \ln|t^2+1| + c$$

$$= \frac{1}{2}\sqrt{x^2-2x-1} + x - \ln\left|2x^2-2x-2+2x\sqrt{x^2-2x-1}\right| +$$

«« انتهت المحاضرة الثالثة »»

مع تمنياتي لكم بالتوفيق والنجاح

إعداد: فاطمة الحسيني